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Geometrical Illustrations of Some Theorems in Number.

By Ellery W. Davis.

The Cartesian coordinates of a point in a plane being denoted by x and y, consider a change to a system whose coordinates, ξ, η, are given by

$$\xi = x \pm y, \quad \eta = y.$$

The transformation preserves all areas and collineations unchanged, together with the order of all directions about any and all points and of all positions along any and all lines.

Let x and y be integers; then also are ξ and η. Moreover, ξ and η are or are not relatively prime according as x and y are.

All that we have said would likewise be true had the transformation been

$$\xi = x, \quad \eta = y \pm x.$$

Furthermore, by a continued repetition of these transformations we can get from any point whose coordinates are relatively prime integers to all other points whose coordinates are relatively prime integers.

For suppose both coordinates positive. Then, by always retaining the smaller coordinate and replacing the larger by the difference of the two, we get at last the point (1, 1); while, by merely reversing the operations that would carry us from a point (x, y) to a point (1, 1), we get from (1, 1) to (x, y). Obvious enough is the modification of this process necessary when one or both coordinates are negative.

Imagine drawn the positive parts of the lines

$$x = 0, 1, 2, 3, \ldots; \quad y = 0, 1, 2, 3, \ldots.$$

The intersections of these lines we call *nodes*. When a node is such that its x and y are relatively prime it is a *filled* node; otherwise the node is *open*.

Marking all the filled nodes by dots gives a diagram used by Prof. Sylvester in demonstrating certain theorems about Farey series. A modification has also been used in the calculation of factor tables.

The actual construction of the diagram is facilitated by taking account of the fact that

$$x = a, \quad y = b \qquad (a > b)$$

gives a filled node if, and only if,

$$x + y = a, \quad y = b$$

also gives one.

Checks are afforded by noticing that also

$$\begin{aligned} x + 2y &= a, \quad y = b, \\ x + 3y &= a, \quad y = b, \\ &\vdots \qquad\;\; \vdots \\ x + ky &= a, \quad y = b, \end{aligned}$$

should give filled nodes when (a, b) is a filled node.

If the reader will examine on our diagram the lines cd $(x = 14)$, cef $(\pm x + y = 14)$, cgh $(\pm x + 2y = 14)$, he will find that they all pass through filled nodes for the same values of y.

A line through the origin and any filled node passes through no other filled node and through no open node between the origin and the filled node. While, since the coordinates of every open node are equimultiples of the coordinates of some filled node, every line through the origin and an open node passes through a filled node between it and the origin. That is, to an eye at the origin all of the filled nodes but none of the open nodes are visible.

Imagine the diagram continued to infinity both positively and negatively. The transformation

$$\xi = x + y, \quad \eta = y$$

then shears the diagram into itself.

In this shearing the parallel lines

$$x = 0, \; x = \pm 1, \; x = \pm 2, \; x = \pm 3, \ldots, \; x = \pm k$$

become the parallels

$$x = y, \; x = y \pm 1, \; x = y \pm 2, \; x = y \pm 3, \ldots, \; x = y \pm k.$$

Now any node on $x = \pm k$ forms with the origin and (0, 1) a triangle of double-area k. Consequently any node on $x = y \pm k$ also forms with the origin and (1, 1) a triangle of double-area k.

Since by a series of such transformations (0, 1) can become any filled node whatever, it follows that parallel to the line joining the origin to any filled node will be pairs of lines of nodes that with the origin and the assumed filled node form triangles of double-area 1, 2, 3, ... , k.

If (a, b) is the fixed assumed node we thus get all the solutions of

$$ay - bx = \pm k, \qquad (k \text{ is integral})$$

with a and b relatively prime. If x and y are also to be relatively prime we confine ourselves to filled nodes on the pairs of lines.

Similarly, taking $(a, -b)$ for the fixed node gives all the integral solutions of

$$ay + bx = \pm k.$$

If a line be drawn joining two nodes but missing the origin, there is a parallel line joining the origin to a filled node. The former line is thus gotten by a series of our transformations from a line $x = \pm k$ and so contains an infinite number of filled nodes. Otherwise there would be a number to which no other number is relatively prime.

It will be seen that the arrangement of the filled nodes on any line

$$x = \text{an integer}$$

is periodic, the period being the product of all the prime factors of the integer. Moreover, the arrangement is the same for all integers having the same prime factors, regardless of the powers to which those factors may be raised.

What has just been said remains true if for "integer" we substitute "set of integers".

Thus the arrangement of filled nodes in the strip

$$x = 1,\ 2,\ 3,\ 4$$

has the period 6; that in the strip

$$x = 1,\ 2,\ 3,\ 4,\ 5,\ 6$$

the period 30; that in the strip

$$x = 1,\ 2, \ldots\ldots, 25$$

the period 210.

Now the line

$$x = \text{any prime}$$

crosses filled nodes except where

$$y = \text{a multiple of that prime.}$$

Consequently a reference to the diagram shows that, having regard to periodicity, numbers greater than 3 cannot be prime unless of one of the forms

$$6n \pm 1.$$

If they are of either of these forms their least factor exceeds 3.

Similarly, numbers greater than 5 cannot be prime unless of one of the forms

$$30n \pm 1,\ 30n \pm 7,\ 30n \pm 11,\ 30n \pm 13;$$

while if of one of these forms their least factors exceed 5.

In general, if p is a prime and P the product of all the primes not greater than it, while

$$k_1,\ k_2,\ k_3,\ \ldots .$$

are the integers not greater than P whose least factor exceeds p; then numbers greater than p cannot be prime unless of one of the forms

$$Pn \pm 1,\ Pn \pm k_1,\ Pn \pm k_2,\ Pn \pm k_3,\ \ldots ,$$

while if of one of these forms their least factors exceed p.

The number of these forms is the totient of P. On our diagram it is the number of filled nodes on $x = P$ below the diagonal $x = y$.

I am indebted to Mr. W. P. Durfee for a very neat illustration of the theorem that

A number equals the sum of the totients of all its divisors.

Join the origin to all the nodes, not above $x = y$ on

$$x = N.$$

Every joining line passes through a filled node on

$$x = \text{some divisor of } N.$$

Conversely, every line from the origin through a filled node, not above $x = y$, on

$$x = \text{some divisor of } N,$$

passes through a node, not above $x = y$, on

$$x = N.$$

As all these lines pass through filled nodes, no two are the same and their number is N.

A similar illustration can be given of the theorem that

The totient of the product of two relatively prime numbers is the product of their totients.

Suppose a, b are the two relatively prime numbers and assume $a < b$.

Moreover, let p, q, r be any numbers respectively not greater than and prime to a, b, and ab.

Then (a, p) is a filled node as are also $(b, q+mb)$ and $(ab, r+nab)$ for all integral values of m and n.

We will prove that for each pair of values of p and q there is one and only one value of r such that the nodes

$$(a, p), \quad (b, q+mb), \quad (ab, r+nab)$$

are collinear.

Collinearity requires that

$$\begin{aligned} nab + r &= a(b-1)(mb+q-p)/(b-a)+p \\ &= ak(mb+q-p)/l+p \end{aligned}$$

if k/l is the fraction $(b-1)/(b-a)$ reduced to its lowest terms.

Since b is prime to $b-a$ there is, for each pair of values of p and q, just one value of m, from 1 to l inclusive, that renders $mb+q-p$ divisible by l. Thus each pair of values of p and q determines r, as an integer, uniquely, values of m outside the range giving new n's but not new r's.

The expression for $nab+r$ is evidently prime to a, and by interchanging a with b, and p with $mb+q$, we can get another expression for $nab+r$ as plainly prime to b. Thus r is prime to ab.

We will now show that different pairs of values of p and q cannot give the same r.

We have

$$mb+q=l(nab+r-p)/ak+p.$$

Here l is prime to a and consequently $r-p$ must be a multiple of a. But r being given, there is just one value of p that will render $r-p$ a multiple of a. Thus to any r there is just one p. Having thus fixed $r-p$, there is just one value of n from 1 to k inclusive that renders $nab+r-p$ divisible by k and so $mb+q$ integral. Thus q is uniquely determined.

Our two propositions established, it follows at once that the number of r's is the product of the number of p's by the number of q's.

On the diagram are represented the collineations when a, b, and ab, are respectively 3, 5, and 15.

As another case, let us take for a, b, and ab, 4, 7, and 28.

The p's are 1 and 3.

The q's are 1, 2, 3, 4, 5, and 6.

The r's are 1, 3, 5, 9, 11, 13, 17, 19, 23, 25, 27.

The collineations are

(4, 1), (7, 1), (28, 1); (4, 1), (7, 46), (28, 361);
(4, 1), (7, 16), (28, 121); (4, 1), (7, 19), (28, 145);
(4, 1), (7, 31), (28, 241); (4, 1), (7, 13), (28, 97);
(4, 3), (7, 15), (28, 99); (4, 3), (7, 18), (28, 123);
(4, 3), (7, 9), (28, 51); (4, 3), (7, 12), (28, 75);
(4, 3), (7, 24), (28, 171); (4, 3), (7, 6), (28, 27).

Since the arrangement of the filled nodes on any line $x = N$ depends only upon the prime factors of N, it follows that the fraction

$$\text{totient-of-}N/N$$

is the same for all numbers having the same prime factors.

For any prime p and its integral powers the fraction is

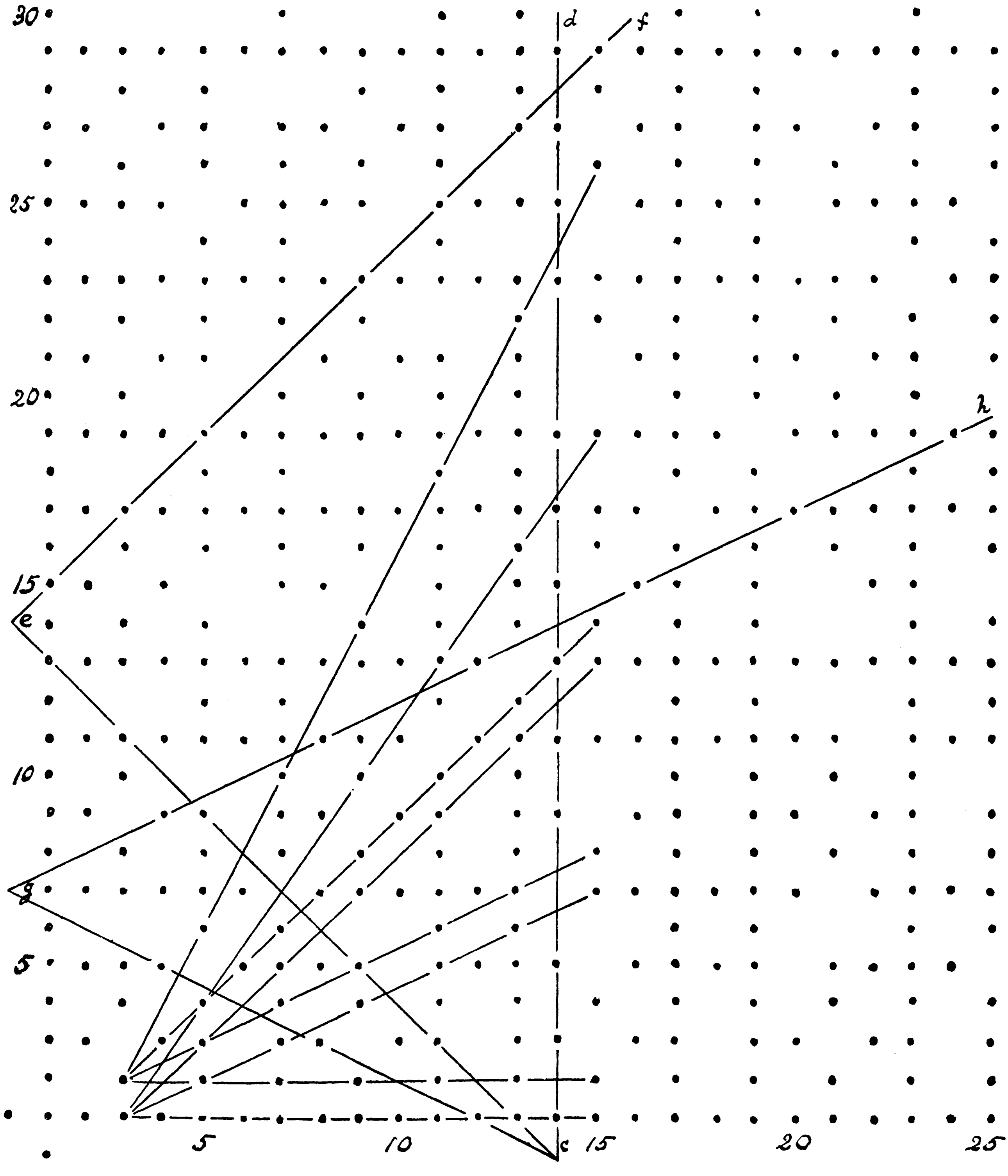
$$(p-1)/p = 1 - 1/p.$$

Consequently, for a number whose prime factors are

$$p,\ q,\ r,\ s, \ldots .$$

the value of the fraction is

$$(1-1/p)(1-1/q)(1-1/r)(1-1/s) \ldots .$$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